INTERNATIONAL JOURNAL OF MULTIDISCIPLINARY COMPREHENSIVE RESEARCH

On Closure and Interior Operators in $\mathcal{N}$ −Space

Maryam M Taresh
Ministry of Education, Babylon Education Directorate, Iraq

* Corresponding Author: **Maryam M Taresh**

Article Info

ISSN (online): 2583-5289
Volume: 05
Issue: 02
March-April 2026
Received: 11-01-2026
Accepted: 09-02-2026
Published: 07-03-2026
Page No: 32-36

Abstract
This paper studies closure structures arising from graphs within the framework of $\mathcal{N}$-space. The concept of neighborhood accumulation vertices is introduced and used to construct the corresponding accumulation graph. Based on this structure, a family of closed graphs and a closure operator are defined. Several fundamental properties of these graphs are investigated, particularly for complete graphs, star graphs, complete bipartite graphs, and null graphs.

DOI: https://doi.org/10.54660/IJMCR.2026.5.2.32-36

Keywords: $\mathcal{N}$-Space, Neighborhood partition system, Accumulation graph, $\mathcal{N}$-Closed graphs, Closure operator

1. Introduction

Graph theory is one of the most important branches of discrete mathematics and has become an essential tool in many areas of science and engineering. Since its early development, graph theory has provided a powerful framework for modeling relationships between objects and studying structural properties of networks. Because of its flexibility and wide applicability, graph theory has attracted considerable attention from researchers and has developed into a rich mathematical discipline with numerous theoretical and practical applications [1, 2, 3]. Topology, on the other hand, is a fundamental area of mathematics concerned with the study of spatial structures and properties that are preserved under continuous transformations. One of the central concepts in topology is the notion of closure, which describes how a set expands to include its limit or accumulation points. Closure operators play a crucial role in understanding the structure of topological spaces and have been widely studied in both classical and generalized topology [4, 5, 7, 8]. In recent years, the interaction between graph theory and topology has led to several interesting developments. Researchers have explored ways of constructing topological-like structures from graphs in order to study combinatorial interpretations of topological notions such as interior, closure, and accumulation. These approaches provide new insights into graph structures and allow the application of topological methods in discrete mathematics. Neighborhood systems in graphs provide a natural tool for developing such structures. By analyzing the relationships between vertices and their neighborhoods, it is possible to define various partition systems that capture important structural properties of graphs. These systems can be used to construct graph-induced spaces in which accumulation and closure concepts arise naturally. In this work, we investigate closure structures generated from neighborhood partition systems within the framework of $\mathcal{N}$-space. In particular, we study the notion of neighborhood accumulation vertices and introduce the associated accumulation graph. Using this concept, a family of closed graphs is defined and several properties of the corresponding closure operator are established. The obtained results describe how these closure structures behave for several well-known classes of graphs. Before presenting the main results, we recall some basic definitions and concepts from graph theory and topology that will be used throughout this paper. Let $\mathcal{W} = \big(\psi(\mathcal{W}), \epsilon(\mathcal{W})\big)$ be a graph, where $\psi(\mathcal{W})$ denotes the set of vertices and $\epsilon(\mathcal{W})$ denotes the set of edges. An edge is an unordered pair of vertices representing a connection between them. A graph is said to be simple if it contains no loops and no multiple edges. A complete graph, denoted by K_n, is a graph in which every pair of distinct vertices is connected by an edge. A complete bipartite graph, denoted by $K_{n,m}$, is a graph whose vertex set can be partitioned into two disjoint subsets such that every vertex in one subset is adjacent to all vertices in the other subset. A null graph is a graph that contains vertices but no edges. A star graph, denoted by S_n, is a special type of tree consisting of a central vertex connected to n leaf vertices.

The neighborhood of r is defined by $\mathcal{N}bd(r) = \{u \in \psi(\mathcal{W}); (u, r) \in \mathrm{E}(\mathcal{W})\}$.[1,2] From the topological point of view, a topology on a set X is a collection τ of subsets of X satisfying the following axioms: the empty set and X belong to τ arbitrary unions of elements of τ belong to τ, and finite intersections of elements of τ belong to τ. The pair (X, τ) is called a topological space.[9, 10] Associated with every topological space is a closure operator. For a subset $A \subseteq X$, the closure of A, usually denoted by $Cl(A)$, is defined as the smallest closed set containing A. Closure operators satisfy several fundamental properties such as extensivity, monotonicity, and idempotency. Inspired by these ideas, closure-type structures can be defined on graph-based systems by using neighborhood relations among vertices. Such constructions allow the study of accumulation behavior and structural properties of graphs from a topological perspective. In this paper, we focus on closure graphs generated from neighborhood systems in $\mathcal{N}$-space and investigate their fundamental properties.

2. Main Results

In this section, we present the main definitions and results concerning closure graphs in $\mathcal{N}$-spaces. Several basic properties of the associated closure operator are also established.

Definition 2.1. Let $\mathcal{W} = \big(\psi(\mathcal{W}), \mathrm{E}(\mathcal{W})\big)$ be a graph and a vertex $r \in \psi(\mathcal{W})$.

1. The common neighborhood set of r is denoted by $\mathcal{CNS}(r)$ and defined by: $\mathcal{CNS}(r) = \{v \in \psi(\mathcal{W}) | \mathcal{N}bd(r) \cap \mathcal{N}bd(v) \neq \emptyset\}$.
2. The disjoint neighborhood set of r is denoted by $\mathcal{DNS}(r)$ and defined by: $\mathcal{DNS}(r) = \{v \in \psi(\mathcal{W}) | \mathcal{N}bd(r) \cap \mathcal{N}bd(v) = \emptyset\}$.

Definition 2.2. Let $\mathcal{W} = \big(\psi(\mathcal{W}), \mathrm{E}(\mathcal{W})\big)$ be a graph, then common neighborhood set system (resp. disjoint neighborhood set system) of a vertex $r \in \psi(\mathcal{W})$ is denoted by $\mathcal{CNSS}(r)$ (resp. $\mathcal{DNSS}(r)$) and defined by: $\mathcal{CNSS}(r) = \{\mathcal{CNS}(r)\}$ (resp. $\mathcal{DNSS}(r) = \{\mathcal{DNS}(r)\}$).

Definition 2.3. Let $\mathcal{W} = \big(\psi(\mathcal{W}), \mathrm{E}(\mathcal{W})\big)$ be a graph the neighborhood partition system of a vertex $r \in \psi(\mathcal{W})$ is denoted by $\mathcal{NPS}(r)$ and defined by: $\mathcal{NPS}(r) = \{\mathcal{CNS}(r), \mathcal{DNS}(r)\}$.

Remark 2.4. Let $\mathcal{W} = \big(\psi(\mathcal{W}), \mathrm{E}(\mathcal{W})\big)$ be a graph the neighborhood partition of a vertex $r \in \psi(\mathcal{W})$ is denoted by $\mathcal{NP}(r)$ such t hat $\mathcal{NP}(r) \in \mathcal{NPS}(r)$.

Definition 2.5. Let $\mathcal{W} = \big(\psi(\mathcal{W}), \mathrm{E}(\mathcal{W})\big)$ be a graph and suppose that $\delta_{\mathcal{N}}: \psi(\mathcal{W}) \longrightarrow \mathrm{P}(\mathrm{P}(\psi(\mathcal{W})))$ is a mapping which assigns for each r in $\psi(\mathcal{W})$ it's neighborhood partition system in $\mathrm{P}\Big(\mathrm{P}\big(\psi(\mathcal{W})\big)\Big)$. The pair $(\mathcal{W}, \delta_{\mathcal{N}})$ is called the $\mathcal{N}$-space. as illustrated in the following example.

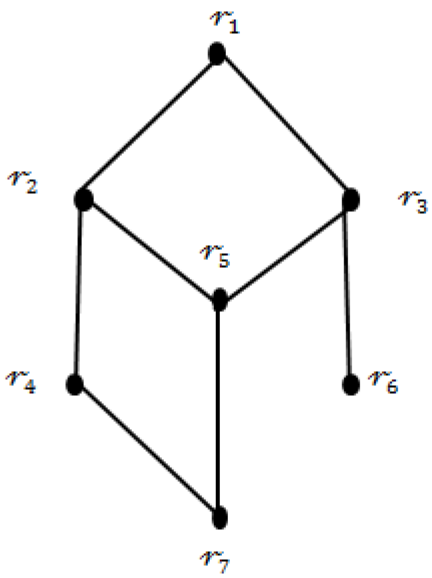

Fig 1:

The mapping $\delta_{\mathcal{N}}$ is given by:

$$\delta_{\mathcal{N}}(r_1) = \big\{\{r_4, r_5, r_6\}, \{r_2, r_3, r_7\}\big\},$$
$$\delta_{\mathcal{N}}(r_2) = \{\{r_3, r_7\}, \{r_1, r_4, r_5, r_6\}\},$$
$$\delta_{\mathcal{N}}(r_3) = \{\{r_2, r_7\}, \{r_1, r_4, r_5, r_6\}\},$$
$$\delta_{\mathcal{N}}(r_4) = \{\{r_1, r_5\}, \{r_2, r_3, r_6, r_7\}\},$$
$$\delta_{\mathcal{N}}(r_5) = \{\{r_1, r_4, r_6\}, \{r_2, r_3, r_7\}\},$$
$$\delta_{\mathcal{N}}(r_6) = \{\{r_1, r_5\}, \{r_2, r_3, r_4, r_7\}\},$$
$$\delta_{\mathcal{N}}(r_7) = \{\{r_2, r_3\}, \{r_1, r_4, r_5, r_6\}\}.$$

Proposition 2.6. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, if $\delta_{\mathcal{N}}$ induced of complete graph $\mathcal{W} = K_n$; $n \geq 3$, then $\delta_{\mathcal{N}}(r) = \{\psi(\mathcal{W}) - \{r\}, \emptyset\}$, for every $r \in \psi(\mathcal{W})$.

Proposition 2.7. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, if $\delta_{\mathcal{N}}$ induced of star graph $\mathcal{W} = K_{1,n}$, with center $\mathcal{C}$ and leavels $L = \{r_1, \dots, r_n\}$

1. $\delta_{\mathcal{N}}(r) = \{\emptyset, L\}$, for the center $\mathcal{C}$.
2. $\delta_{\mathcal{N}}(r) = \{L - \{r_i\}, \{\mathcal{C}\}\}$ foe any leaf r_i

Proposition 2.8. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, if $\delta_{\mathcal{N}}$ induced of complete bipartite graph $\mathcal{W} = K_{m,n}$, with bipartition $\psi = \mathrm{X} \cup \mathrm{Y}$ where $|\mathrm{X}| = m$ and $|\mathrm{Y}| = n$. For any vertex $r \in \psi$, the mapping $\delta_{\mathcal{N}}$ satisfy the following:

1. $\delta_{\mathcal{N}}(r) = \{X - \{r\}, Y\}$ for any $r \in X$,
2. $\delta_{\mathcal{N}}(r) = \{Y - \{r\}, X\}$ for any $r \in Y$.

Proposition 2.9. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, if $\delta_{\mathcal{N}}$ induced of null graph, then

$$\delta_{\mathcal{N}}(r) = \{\emptyset, \psi(\mathcal{W}) - \{r\}\}, \text{ for every } r \in \psi(\mathcal{W}).$$

Definition 2.10. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space and let $\mathcal{G} \subseteq \mathcal{W}$. A vertex r in $\psi(\mathcal{W})$ is called a neighborhood accumulation vertex of $\mathcal{G}$ if every neighborhood partition of r contains at least one vertex of $\mathcal{G}$ different from r. That is r is a $\mathcal{N}AV$-point of $\mathcal{G}$ if $\forall\, \mathcal{NP}(r), \mathcal{NP}(r) \cap (\psi(\mathcal{G}) - \{r\}) \neq \phi$ The set of all neighborhood accumulation vertices of $\mathcal{G}$ is called $\mathcal{N}$- accumulation graph of $\mathcal{G}$ and is denoted by $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc}$, where : $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \{r \in \psi(\mathcal{W}); \forall\, \mathcal{NP}(r), \mathcal{NP}(r) \cap (\psi(\mathcal{G}) - \{r\}) \neq \phi\}$.

Definition 2.11. An $\mathcal{N}$-space $(\mathcal{W}, \delta_{\mathcal{N}})$, and let $\mathcal{G}$ be a subgraph of $\mathcal{W}$. The graph $\mathcal{G}$ is said to be $\mathcal{N}$-closed if it contains all neighborhood accumulation vertices of $\mathcal{G}$. The family $\Psi_{\delta_{\mathcal{N}}}$ of all $\mathcal{N}$-closed graphs of an $\mathcal{N}$-space is defined by:

$$\Psi_{\delta_{\mathcal{N}}} = \{\psi(\mathcal{G}) \subseteq \psi(\mathcal{W}) \mid [\psi(\mathcal{G})]_{\mathcal{N}}^{acc} \subseteq \psi(\mathcal{G})\}.$$

Theorem 2.12. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$ − space and let $\Psi_{\delta_{\mathcal{N}}}$ be a family of all $\mathcal{N}$-closed graphs, then :

1. $\emptyset$ and $\mathcal{W} \in \Psi_{\delta_{\mathcal{N}}}$.
2. For any subfamily $\{\mathcal{G}_i\}_{i\in I} \subseteq \Psi_{\delta_{\mathcal{N}}}$ then $\cap_{i\in I}\{\mathcal{G}_i\} \in \Psi_{\delta_{\mathcal{N}}}$.

Proof

1. Clear.

2. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be an $\mathcal{N}$-space such that $\mathcal{J} \subseteq \mathcal{W}$ and $\psi(\mathcal{J}) = \cap_i \psi(\mathcal{G}_i)$; $i \in I$, the intersection of the $\mathcal{N}$-closed $\mathcal{G}_i \subseteq \mathcal{W}, i \in I$. Hence $\psi(\mathcal{J}) \subseteq \psi(\mathcal{G}_i)$ for all $i \in I$ which implies $[\psi(\mathcal{J})]_{\mathcal{N}}^{acc} \subseteq [\psi(\mathcal{G}_i)]_{\mathcal{N}}^{acc}$ for all $i \in I$. But $[\psi(\mathcal{G}_i)]_{\mathcal{N}}^{acc} \subseteq \psi(\mathcal{G}_i)$ for all $i \in I$ since $\mathcal{G}_i$ is $\mathcal{N}$-closed and so $[\psi(\mathcal{J})]_{\mathcal{N}}^{acc} \subseteq \psi(\mathcal{G}_i)$ for all $i \in I$ thus, $[\psi(\mathcal{J})]_{\mathcal{N}}^{acc} \subseteq \cap_i \left(\psi(\mathcal{G}_i)\right) = \psi(\mathcal{J})$, hence $\mathcal{J}$ is $\mathcal{N}$-closed.

If $\cap_i \left(\psi(\mathcal{G}_i)\right) = \emptyset$ by (1) above we get $\cap_i \left(\psi(\mathcal{G}_i)\right)$ is $\mathcal{N}$-closed.

If $\cap_i \left(\psi(\mathcal{G}_i)\right) = \mathcal{W}$ by (1) above we get $\cap_i \left(\psi(\mathcal{G}_i)\right)$ is $\mathcal{N}$-closed.

Theorem 2.15. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, then every singleton subgraph $\mathcal{G}$ of $\mathcal{W}$ is $\mathcal{N}$-closed.

Proof: Let $\mathcal{G} \subseteq \mathcal{W}$ be singleton subgraph, where $\mathcal{G} = \{t\}$. We now observe that $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = [\{t\}]_{\mathcal{N}}^{acc} = \phi$, because, suppose that $[\{t\}]_{\mathcal{N}}^{acc} \neq \phi \implies \exists\, r \in \psi(\mathcal{W})$ such that $r \in [\{t\}]_{\mathcal{N}}^{acc} \implies \mathcal{NP}(r) \cap (\{t\} - \{r\}) \neq \phi, \forall\, \mathcal{NP}(r) \implies t \neq r$ and $r \in \mathcal{NP}(t)$, $\forall\, \mathcal{NP}(t) \implies r \in \mathcal{CNS}(t)$ and $r \in \mathcal{DNS}(t)$ and this a contradiction then $r \notin [\{t\}]_{\mathcal{N}}^{acc} \implies [\{t\}]_{\mathcal{N}}^{acc} = \phi$. We get $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \phi$. And hence $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} \subseteq \psi(\mathcal{G})$. Therefore $\mathcal{G}$ is $\mathcal{N}$-closed.

Theorem 2.16. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space. If $\mathcal{W}$ is complete graph then every subgraph $\mathcal{G}$ of $\mathcal{W}$ is $\mathcal{N}$-closed.

Proof: Let $\mathcal{W}$ be a complete graph and let $\mathcal{G} \subseteq \mathcal{W}$ be any subgraph. We now observe that $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \phi$, because, suppose that $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} \neq \phi \implies \exists\, r \in \psi(\mathcal{W})$ and $r \in [\psi(\mathcal{H})]_{\mathcal{N}}^{acc} \implies \mathcal{NP}(r) \cap (\psi(\mathcal{G}) - \{r\}) \neq \phi, \forall \mathcal{NP}(r) \implies \mathcal{NP}(r)$ contains at least one vertex of $\mathcal{G} \subseteq \mathcal{W}, \forall\, \mathcal{NP}(r) \implies \mathcal{NP}(r) \neq \phi, \forall\, \mathcal{NP}(r)$. Since $\mathcal{DNS}(r) = \mathcal{NP}(r) \implies \mathcal{DNS}(r) \neq \phi$. Then there exists at least one vertex say w in $\mathcal{DNS}(r)$ and this a contradiction with Proposition 2.6. Hence $[\psi(\mathcal{H})]_{\mathcal{N}}^{acc} = \phi$. And we get $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} \subseteq \psi(\mathcal{G})$. Therefore $\mathcal{G}$ is $\mathcal{N}$-closed.

Corollary 2.17. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, then if $\mathcal{W}$ is complete graph then $\Psi_{\delta_{\mathcal{N}}} = P\big(\psi(\mathcal{W})\big)$.

Theorem 2.18. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space. If $\mathcal{W} = K_{m,n}$ be a complete bipartite graph having bipartition (ψ_1, ψ_2) then every sub graph $\mathcal{G}$ of $\mathcal{W}$ we have.

1. If $\mathcal{G} \subseteq \psi_1$ or ψ_2 then $\mathcal{G}$ is $\mathcal{N}$-closed.
2. If $\mathcal{G} \subsetneq \mathcal{W}$ and contains at least two vertices from (ψ_1 and ψ_2) then $\mathcal{G}$ is not $\mathcal{N}$-closed.

Proof: Let $\mathcal{W} = K_{m,n}$ be a complete bipartite graph.

1. Let $\mathcal{G} \subseteq \mathcal{W}$ such that $\mathcal{G} \subseteq \psi_1$. We now observe that $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \phi$. Because, Let $r \in \psi(\mathcal{W})$ to prove that $r \notin [\psi(\mathcal{G})]_{\mathcal{N}}^{acc}$. By definition of complete bipartite graph we get $\psi_1 \cap \psi_2 = \phi$. And $\psi(\mathcal{G}) \cap \psi_2 = \phi$. (Since $\psi(\mathcal{G}) \subseteq \psi_1$). And hence $(\psi(\mathcal{G}) - \{r\}) \cap (\psi_2 - \{r\}) = \phi$. By Proposition 2.8 we get $[\forall\, r \in \psi(\mathcal{W})$ then $\delta_{\mathcal{N}}(r) = \{\psi_1 - \{r\}, \psi_2 - \{r\}\}]$. Then $\exists\, \mathcal{NP}(v) = \psi_2 - \{r\}$ such that $\mathcal{NP}(r) \cap (\psi(\mathcal{G}) - \{r\}) = \phi$. Then $r \notin [\psi(\mathcal{G})]_{\mathcal{N}}^{acc} \implies [\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \phi$. And by the same way if $\psi(\mathcal{G}) \subseteq \psi_2$. We get for all $\mathcal{G} \subseteq \psi_1$ or ψ_2, $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \phi \implies [\psi(\mathcal{G})]_{\mathcal{N}}^{acc} \subseteq \psi(\mathcal{G})$. Then $\mathcal{G}$ is $\mathcal{N}$-closed.
2. Let $\psi(\mathcal{G})$ contains at least two vertices from ψ_1 say a, b and contains at least two vertices from ψ_2 say c, d. $\mathcal{N}$ow to prove that $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \psi(\mathcal{W})$. Let $r \in \psi(\mathcal{W})$. To prove $r \in [\psi(\mathcal{G})]_{\mathcal{N}}^{acc}$. From Proposition 2.8, we get $\forall\, r \in \psi(\mathcal{W})$ then $\delta_{\mathcal{N}}(r) = \{\psi_1 - \{r\}, \psi_2 - \{r\}\}$. If $r \neq a, b, c$ or d. Then $(\psi_1 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\})$ contains at least a, b. Then $(\psi_1 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\}) \neq \phi$. And $(\psi_2 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\})$ contains at least c, d. Then $(\psi_2 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\}) \neq \phi$. And $\mathcal{NP}(r) = \psi_1 - \{r\}$ or $\psi_2 - \{r\}$ hence $r \in [\psi(\mathcal{G})]_{\mathcal{N}}^{acc}$. if $r = a, b, c$ or d. Then if $r = a$ Then $(\psi_1 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\})$ contains at least b. And if $r = b$ then $(\psi_1 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\})$ contains at least a. Then $(\psi_1 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\}) \neq \phi$. Also, if $r = c$ Then $(\psi_2 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\})$ contains at least d. And if $r = d$ then $(\psi_2 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\})$ contains at least c. Then $(\psi_2 - \{r\}) \cap (\psi(\mathcal{G}) - \{r\}) \neq \phi$. Then $r \in [\psi(\mathcal{G})]_{\mathcal{N}}^{acc}$. And hence $\forall\, r \in \psi(\mathcal{W}) \implies r \in [\psi(\mathcal{G})]_{\mathcal{N}}^{acc}$. Then $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \psi(\mathcal{W})$. So for all $\mathcal{G} \subsetneq \mathcal{W}$ and contains at least two vertices from (ψ_1 and ψ_2) $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} = \psi(\mathcal{W}) \nsubseteq \psi(\mathcal{G})$ and $\mathcal{G}$ is not $\mathcal{N}$-closed.

Theorem 2.19. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, and $\mathcal{G} \subseteq \mathcal{W}$ is $\mathcal{N}$-closed graph, then every graph contained in $\mathcal{G}$ and containing $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc}$ is $\mathcal{N}$-closed.

Proof: Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be an $\mathcal{N}$-space and $\mathcal{G}, \mathcal{J} \subseteq \mathcal{W}$ such that $\mathcal{G}$ is $\mathcal{N}$-closed graph and $[\psi(\mathcal{G})]_{\mathcal{N}}^{acc} \subseteq \psi(\mathcal{J}) \subseteq \psi(\mathcal{G})$. Since $\psi(\mathcal{J}) \subseteq \psi(\mathcal{G})$ then $[\psi(\mathcal{J})]_{\mathcal{N}}^{acc} \subseteq [\psi(\mathcal{G})]_{\mathcal{N}}^{acc}$ and so $[\psi(\mathcal{J})]_{\mathcal{N}}^{acc} \subseteq \psi(\mathcal{J})$, there for $\mathcal{J}$ is $\mathcal{N}$-closed.

Corollary 2.20. The $\mathcal{N}$- accumulation graph from an $\mathcal{N}$-closed graph is $\mathcal{N}$-closed.

Definition 2.21. Let $\mathcal{G}$ be a subgraph of a $\mathcal{N}$-space $(\mathcal{W}, \delta_{\mathcal{N}})$. The intersection of all $\mathcal{N}$-closed graphs containing $\mathcal{G}$ is called $\mathcal{N}$-closure of $\mathcal{G}$ and is denoted by $Clr_{\mathcal{N}}(\psi(\mathcal{G}))$, i.e.: $Clr_{\mathcal{N}}(\psi(\mathcal{G})) = \cap \{\psi(\mathcal{J}) \in \Psi_{\delta_{\mathcal{N}}}; \psi(\mathcal{G}) \subseteq \psi(\mathcal{J})\}$.
The operator $Clr_{\mathcal{N}}: P(\psi(\mathcal{W})) \longrightarrow P(\psi(\mathcal{W}))$ is called $\mathcal{N}$-closure operator.

Remark 2.22. By Theorem 2.12(ii), $Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ is $\mathcal{N}$-closed graph for all $\mathcal{G} \subseteq \mathcal{W}$. Moreover, it is the smallest $\mathcal{N}$-closed graph containing $\psi(\mathcal{G})$. $\mathcal{G}$ is $\mathcal{N}$-closed if and only if $\psi(\mathcal{G}) = Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ and in particular, $Clr_{\mathcal{N}}\left(Clr_{\mathcal{N}}(\psi(\mathcal{G}))\right) = Clr_{\mathcal{N}}(\psi(\mathcal{G}))$.

Proposition 2.24. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space and $\mathcal{G} \subseteq \mathcal{W}$, then $\psi(\mathcal{G}) \cup [\psi(\mathcal{G})]^{acc}_{\mathcal{N}} \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$.

Proof: Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be an $\mathcal{N}$-space and $\mathcal{G} \subseteq \mathcal{W}$. Since $\psi(\mathcal{G}) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ then $[\psi(\mathcal{G})]^{acc}_{\mathcal{N}} \subseteq [Clr_{\mathcal{N}}(\psi(\mathcal{G}))]^{acc}_{\mathcal{N}}$. But $[Clr_{\mathcal{N}}(\psi(\mathcal{G}))]^{acc}_{\mathcal{N}} \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ because $Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ is $\mathcal{N}$-closed and so $[\psi(\mathcal{G})]^{acc}_{\mathcal{N}} \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$. Accordingly, $\psi(\mathcal{G}) \cup [\psi(\mathcal{G})]^{acc}_{\mathcal{N}} \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$.

Remark 2.25. If $(\mathcal{W}, \delta_{\mathcal{N}})$ is an $\mathcal{N}$-space $\mathcal{G} \subseteq \mathcal{W}$, then the relation $\psi(\mathcal{G}) \cup [\psi(\mathcal{G})]^{acc}_{\mathcal{N}} = Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ is not necessarily true.

Proposition 2.26. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be an $\mathcal{N}$-space, and $\mathcal{G} \subseteq \mathcal{W}$ if $\mathcal{G}$ is $\mathcal{N}$-closed then $\psi(\mathcal{G}) \cup [\psi(\mathcal{G})]^{acc}_{\mathcal{N}} = Clr_{\mathcal{N}}(\psi(\mathcal{G}))$.

Proof: Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be an $\mathcal{N}$-space and $\mathcal{G} \subseteq \mathcal{W}$ since $\mathcal{G}$ is $\mathcal{N}$-closed from Definition 2.11 $[\psi(\mathcal{G})]^{acc}_{\mathcal{N}} \subseteq \psi(\mathcal{G})$ thus $\psi(\mathcal{G}) \cup [\psi(\mathcal{G})]^{acc}_{\mathcal{N}} = \psi(\mathcal{G})$ and from Remark 2.22. Hence $\psi(\mathcal{G}) \cup [\psi(\mathcal{G})]^{acc}_{\mathcal{N}} = Clr_{\mathcal{N}}(\psi(\mathcal{G}))$.

Corollary 2.27. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, if $\mathcal{W}$ is complete graph, then $\psi(\mathcal{G}) \cup [\psi(\mathcal{G})]^{acc}_{\mathcal{N}} = Clr_{\mathcal{N}}(\psi(\mathcal{G}))$, for every $\mathcal{G} \subseteq \mathcal{W}$.

Proposition 2.28. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, then the $\mathcal{N}$-closure operator $Clr_{\mathcal{N}}: P(\psi(\mathcal{W})) \longrightarrow P(\psi(\mathcal{W}))$ possesses the following properties for all $\mathcal{G}, \mathcal{J} \subseteq \mathcal{W}$:

a. $Clr_{\mathcal{N}}(\phi) = \phi$,
b. $Clr_{\mathcal{N}}(\psi(\mathcal{W})) = \psi(\mathcal{W})$,
c. $\psi(\mathcal{G}) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$,
d. If $\mathcal{G} \subseteq \mathcal{J}$ then $Clr_{\mathcal{N}}(\psi(\mathcal{G})) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{J}))$,
e. $Clr_{\mathcal{N}}(Clr_{\mathcal{N}}(\psi(\mathcal{G})) = Clr_{\mathcal{N}}(\psi(\mathcal{G}))$,
f. $Clr_{\mathcal{N}}(\psi(\mathcal{G}) \cap \psi(\mathcal{J})) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G})) \cap Clr_{\mathcal{N}}(\psi(\mathcal{J}))$ and
g. $Clr_{\mathcal{N}}(\psi(\mathcal{G}) \cup \psi(\mathcal{J})) \supseteq Clr_{\mathcal{N}}(\psi(\mathcal{G})) \cup Clr_{\mathcal{N}}(\psi(\mathcal{J}))$.

Proof

a. Since $[\phi]^{acc}_{\mathcal{N}} = \phi \Longrightarrow [\phi]^{acc}_{\mathcal{N}} \subseteq \phi \Longrightarrow \phi$ is $\mathcal{N}$-closed and containing ϕ, since $Clr_{\mathcal{N}}(\phi)$ is the smallest $\mathcal{N}$-closed graph containing ϕ, then we get $Clr_{\mathcal{N}}(\phi) = \phi$.
b. Since $\psi(\mathcal{W})$ is $\mathcal{N}$-closed and containing $\psi(\mathcal{W})$ and since $Clr_{\mathcal{N}}(\psi(\mathcal{W}))$ is the smallest $\mathcal{N}$-closed graph containing $\psi(\mathcal{W})$, then we get $Clr_{\mathcal{N}}(\psi(\mathcal{W})) = \psi(\mathcal{W})$.
c. By Proposition 2.24, we get $\psi(\mathcal{G}) \cup [\psi(\mathcal{G})]^{acc}_{\mathcal{N}} \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G})) \Longrightarrow \psi(\mathcal{G}) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$.
d. Let $\mathcal{G} \subseteq \mathcal{J}$, then for all $\mathcal{N}$-closed contains $\mathcal{J}$ is also containing $\mathcal{G}$, thus the intersection of all $\mathcal{N}$-closed containing $\mathcal{G}$ is sub set of the intersection of all $\mathcal{N}$-closed containing $\mathcal{J}$,there for $Clr_{\mathcal{N}}(\psi(\mathcal{G})) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{J}))$.
e. Since $Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ is $\mathcal{N}$-closed graph for all $\mathcal{G} \subseteq \mathcal{W}$. Hence $Clr_{\mathcal{N}}(Clr_{\mathcal{N}}(\psi(\mathcal{G})) = Clr_{\mathcal{N}}(\psi(\mathcal{G}))$.
f. Let $\Psi_{(\mathcal{G}\cap\mathcal{J})}$ (resp. $\Psi_{\mathcal{G}}$ and $\Psi_{\mathcal{J}}$) be the family of all $\mathcal{N}$-closed containing $(\mathcal{G} \cap \mathcal{J})$ (resp. $\mathcal{G}$ and $\mathcal{J}$). i.e. $\Psi_{(\mathcal{G}\cap\mathcal{J})} = \{A \in \Psi_{\delta_{\mathcal{N}}}; (\mathcal{G} \cap \mathcal{J}) \subseteq A\}, \Psi_{\mathcal{G}} = \{A \in \Psi_{\delta_{\mathcal{N}}}; \mathcal{G} \subseteq A\}, \Psi_{\mathcal{J}} = \{A \in \Psi_{\delta_{\mathcal{N}}}; \mathcal{J} \subseteq A\}$. Since $(\mathcal{G} \cap \mathcal{J}) \subseteq \mathcal{G}$ and $(\mathcal{G} \cap \mathcal{J}) \subseteq \mathcal{J} \Longrightarrow \Psi_{\mathcal{G}} \subseteq \Psi_{(\mathcal{G}\cap\mathcal{J})}$ and $\Psi_{\mathcal{J}} \subseteq \Psi_{(\mathcal{G}\cap\mathcal{J})} \Longrightarrow \cap \Psi_{\mathcal{G}\cap\mathcal{J}} \subseteq \cap \Psi_{\mathcal{G}}$ and $\cap \Psi_{\mathcal{G}\cap\mathcal{J}} \subseteq \cap \Psi_{\mathcal{J}}$. Since $Clr_{\mathcal{N}}(\psi(\mathcal{G})) = \cap \{\psi(A) \in \Psi_{\delta_{\mathcal{N}}}; \psi(\mathcal{G}) \subseteq \psi(A)\}$. Then we get, $Clr_{\mathcal{N}}(\psi(\mathcal{G}) \cap \psi(\mathcal{J})) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ and $Clr_{\mathcal{N}}(\psi(\mathcal{G}) \cap \psi(\mathcal{J})) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{J})) \Longrightarrow Clr_{\mathcal{N}}(\psi(\mathcal{G}) \cap \psi(\mathcal{J})) \subseteq Clr_{\mathcal{N}}(\psi(\mathcal{G})) \cap Clr_{\mathcal{N}}(\psi(\mathcal{J}))$.
g. Let $\Psi_{(\mathcal{G}\cup\mathcal{J})}$ (resp. $\Psi_{\mathcal{G}}$ and $\Psi_{\mathcal{J}}$) be the family of all $\mathcal{N}$-closed containing $(\mathcal{G} \cup \mathcal{J})$ (resp. $\mathcal{G}$ and $\mathcal{J}$). i.e. $\Psi_{(\mathcal{G}\cup\mathcal{J})} = \{A \in \Psi_{\delta_{\mathcal{N}}}; (\mathcal{G} \cup \mathcal{J}) \subseteq A\}, \Psi_{\mathcal{G}} = \{A \in \Psi_{\delta_{\mathcal{N}}}; \mathcal{G} \subseteq A\}, \Psi_{\mathcal{J}} = \{A \in \Psi_{\delta_{\mathcal{N}}}; \mathcal{J} \subseteq A\}$. Since $(\mathcal{G} \cup \mathcal{J}) \supseteq \mathcal{G}$ and $(\mathcal{G} \cup \mathcal{J}) \supseteq \mathcal{J} \Longrightarrow \Psi_{\mathcal{G}} \supseteq \Psi_{(\mathcal{G}\cup\mathcal{J})}$ and $\Psi_{\mathcal{J}} \supseteq \Psi_{(\mathcal{G}\cup\mathcal{J})} \Longrightarrow \cap \Psi_{\mathcal{G}\cup\mathcal{J}} \supseteq \cap \Psi_{\mathcal{G}}$ and $\cap \Psi_{\mathcal{G}\cup\mathcal{J}} \supseteq \cap \Psi_{\mathcal{J}}$. Since $Clr_{\mathcal{N}}(\psi(\mathcal{G})) = \cap \{\psi(A) \in \Psi_{\delta_{\mathcal{N}}}; \psi(\mathcal{G}) \subseteq \psi(A)\}$. Then we get, $Clr_{\mathcal{N}}(\psi(\mathcal{G}) \cup \psi(\mathcal{J})) \supseteq Clr_{\mathcal{N}}(\psi(\mathcal{G}))$ and $Clr_{\mathcal{N}}(\psi(\mathcal{G}) \cup \psi(\mathcal{J})) \supseteq Clr_{\mathcal{N}}(\psi(\mathcal{J})) \Longrightarrow Clr_{\mathcal{N}}(\psi(\mathcal{G}) \cup \psi(\mathcal{J})) \supseteq Clr_{\mathcal{N}}(\psi(\mathcal{G})) \cup Clr_{\mathcal{N}}(\psi(\mathcal{J}))$.

Definition 2.29. The complement of an $\mathcal{N}$-closed graph with respect to the $\mathcal{N}$-space $(\mathcal{W}, \delta_{\mathcal{N}})$ which it is contained is called $\mathcal{N}$-open graph the family $\tau_{\delta_{\mathcal{N}}}$ of all $\mathcal{N}$-open graphs is defined by:

$$\tau_{\delta_{\mathcal{N}}} = \{\psi(\mathcal{J}) \subseteq \psi(\mathcal{W}) :: \psi(\mathcal{J}) = \psi(\mathcal{W}) \setminus \psi(\mathcal{G}) \text{ where } \psi(\mathcal{G}) \in \Psi_{\delta_{\mathcal{N}}}\}.$$

Theorem 2.30. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space and let $\tau_{\delta_{\mathcal{N}}}$ be a family of all $\mathcal{N}$-open graphs, then :

1. $\emptyset$ and $\mathcal{W} \in \tau_{\delta_{\mathcal{N}}}$.
2. The union of any family of $\mathcal{N}$-open graphs is $\mathcal{N}$-open.

Proof

1. Clear.
2. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space such that $\mathcal{J} \subseteq \mathcal{W}$ and $\psi(\mathcal{G}) = \cup_i \psi(\mathcal{J}_i)$ be the union of $\mathcal{N}$-open graphs $\mathcal{J}_i \subseteq \wp, i \in I$. Hence $\psi(\mathcal{W}) \setminus \psi(\mathcal{G}) = \psi(\mathcal{W}) \setminus \cup_i \psi(\mathcal{J}_i) = \cap_i [\psi(\mathcal{W}) \setminus \psi(\mathcal{J}_i)]$. Since $\mathcal{J}_i, i \in I$, is $\mathcal{N}$-closed graph hence $\psi(\mathcal{W}) \setminus \psi(\mathcal{G})$ is $\mathcal{N}$-closed and therefore $\mathcal{G}$ is $\mathcal{N}$-open.

Corollary 2.31. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, then the family $\tau_{\delta_{\mathcal{N}}}$ of all $\mathcal{N}$-open graphs forms are supra topology on $\mathcal{W}$.

Theorem 2.32. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, and $\mathcal{I} \subseteq \mathcal{W}$, then $\mathcal{J}$ is $\mathcal{N}$-open if and only if it contains at least one neighborhood partition of each of its vertices.

Proof

Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space and $\mathcal{J}$ be an $\mathcal{N}$-open graph contained in $\mathcal{W}$ and $r \in \psi(\mathcal{J})$. suppose that for each neighborhood partition of r, $\mathcal{NP}(r)$, we have $\mathcal{NP}(r) \nsubseteq \psi(\mathcal{J})$, thus for each $\mathcal{NP}(r)$, $\mathcal{NP}(r) \cap [\psi(\mathcal{W}) - \psi(\mathcal{J})] \neq \phi$ which implies $r \in [\psi(\mathcal{W}) - \psi(\mathcal{J})]'_{\mathcal{N}}$. But $\mathcal{W} - \mathcal{J}$ is $\mathcal{N}$-closed since $\mathcal{J}$ is $\mathcal{N}$-open and so $[\psi(\mathcal{W}) - \psi(\mathcal{J})]'_{\mathcal{N}} \subseteq [\psi(\mathcal{W}) - \psi(\mathcal{J})]$ and hence $r \in [\psi(\mathcal{W}) - \psi(\mathcal{J})]$. Therefore $r \notin \psi(\mathcal{J})$ which contradicts with $r \in \psi(\mathcal{J})$ and consequently if $\mathcal{J} \subseteq \mathcal{W}$ is $\mathcal{N}$-open and $r \in \psi(\mathcal{J})$, then there exist at least one neighborhood partition of r which is contained in $\psi(\mathcal{J})$. conversely, let $\mathcal{J}$ contains at least one neighborhood partition of each of its vertices, i.e. for all $r \in \psi(\mathcal{J})$ there exists $\mathcal{NP}(r)$ such that $\mathcal{NP}(r), \subseteq \psi(\mathcal{J})$. Let $h \in [\psi(\mathcal{W}) - \psi(\mathcal{J})]'_{\mathcal{N}}$ then $h \notin \psi(\mathcal{J})$. If $h \in \psi(\mathcal{J})$ there would be a neighborhood partition of h, $\mathcal{NP}(h)$, such that $\mathcal{NP}(h) \subseteq \psi(\mathcal{J})$ and this would imply that $\mathcal{NP}(h) \cap [\psi(\mathcal{W}) - \psi(\mathcal{J})] = \phi$, thus $h \notin [\psi(\mathcal{W}) - \psi(\mathcal{J})]'_{\mathcal{N}}$ which is impossible. Accordingly, $h \in [\psi(\mathcal{W}) - \psi(\mathcal{J})]$ and so $[\psi(\mathcal{W}) - \psi(\mathcal{J})]'_{\mathcal{N}} \subseteq [\psi(\mathcal{W}) - \psi(\mathcal{J})]$ which implies $\mathcal{W} - \mathcal{J}$ is $\mathcal{N}$-closed and hence $\mathcal{J}$ is $\mathcal{N}$-open.

Definition 2.33. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be an $\mathcal{N}$-space and $\mathcal{J} \subseteq \mathcal{W}$, then the union of all $\mathcal{N}$-open graphs contained in $\mathcal{J}$ is called the $\mathcal{N}$-interior of $\mathcal{J}$ and denoted by $int_{\mathcal{N}}(\psi(\mathcal{J}))$, where $int_{\mathcal{N}}(\psi(\mathcal{J})) = \cup \{\psi(\mathcal{G}) \in \tau_{\delta_{\mathcal{N}}} : \psi(\mathcal{G}) \subseteq \psi(\mathcal{J})\}$.
The operator $int_{\mathcal{N}}: P(\psi(\mathcal{J})) \longrightarrow P(\psi(\mathcal{J}))$ is called the $\mathcal{N}$-interior operator.

Proposition 2.34. Let $(\mathcal{W}, \delta_{\mathcal{N}})$ be a $\mathcal{N}$-space, then the $\mathcal{N}$-interior operator $int_{\mathcal{N}}: P(\psi(\mathcal{J})) \longrightarrow P(\psi(\mathcal{J}))$ satisfies the following properties for all $\mathcal{J}, \mathcal{G} \subseteq \mathcal{W}$:

a. $int_{\mathcal{N}}(\phi) = \phi$,
b. $int_{\mathcal{N}}(\psi(\mathcal{W})) = \psi(\mathcal{W})$,
c. $int_{\mathcal{N}}(\psi(\mathcal{J})) \subseteq \psi(\mathcal{J})$,
d. If $\mathcal{J} \subseteq \mathcal{G}$ then $int_{\mathcal{N}}(\psi(\mathcal{J})) \subseteq int_{\mathcal{N}}(\psi(\mathcal{G}))$,
e. $int_{\mathcal{N}}(int_{\mathcal{N}}(\psi(\mathcal{J})) = int_{\mathcal{N}}(\psi(\mathcal{J}))$,
f. $int_{\mathcal{N}}(\psi(\mathcal{J}) \cap \psi(\mathcal{G})) \subseteq int_{\mathcal{N}}(\psi(\mathcal{J})) \cap int_{\mathcal{N}}(\psi(\mathcal{G}))$ and
g. $int_{\mathcal{N}}(\psi(\mathcal{I}) \cup \psi(\mathcal{G})) \supseteq int_{\mathcal{N}}(\psi(\mathcal{J})) \cup int_{\mathcal{N}}(\psi(\mathcal{G}))$.

Proof: Straightforward.

References

1. Harary F. Graph Theory. Reading (MA): Addison-Wesley; 1969.
2. West DB. Introduction to Graph Theory. 2nd ed. Upper Saddle River (NJ): Prentice Hall; 2001.
3. Bondy JA, Murty USR. Graph Theory. New York: Springer; 2008.
4. Fandy HH, Al'Dzhabri KS. Admixture vertex edges system and A-space. J Discrete Math Sci Cryptogr. 2025;28(2):469–474.
5. Fandy HH, Al'Dzhabri KS. Supra topological space using admixture vertex edges system. AIP Conf Proc. 2026;3393:070090.
6. Al'Dzhabri KS, Rodionov IV. On some formulas in the problem of enumeration of finite labeled topologies. Discrete Math Algorithms Appl. 2023;15(2):2250075. Available from: https://doi.org/10.1142/S1793830922500756
7. Al'Dzhabri KS. Enumeration of connected components of acyclic digraph. J Discrete Math Sci Cryptogr. 2021;24(7):2047–2058.
8. Subbaih SP. A study of graph theory: topology, Steiner domination and semigraph concepts [PhD thesis]. Madurai (India): Madurai Kamaraj University; 2007.
9. Karunakaran K. Topics in graph theory – topological approach [PhD thesis]. Kerala (India): University of Kerala; 2007.
10. Munkres JR. Topology. 2nd ed. Upper Saddle River (NJ): Prentice Hall; 2000.

How to Cite This Article

Taresh MM. On closure and interior operators in N-space. Int J Multidiscip Compr Res. 2026 Mar–Apr;5(2):32–36. doi:10.54660/IJMCR.2026.5.2.32-36.

Creative Commons (CC) License